DARBOUX TRANSFORMATION FOR DIRAC EQUATIONS WITH $(1+1)$ POTENTIALS

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Abstract. We study the Darboux transformation (DT) for Dirac equations with $(1+1)$ potentials. Exact solutions for the adiabatic external field are constructed. The connection between the exactly soluble Dirac (1+1) potentials and the soliton solutions of the Davey–Stewartson equations is discussed.

1. INTRODUCTION.

The Darboux transformation (DT) is a convenien way to construct a rich set of integrable potentials of the steady–state Shrödinger equation in the single dimensional case [1]. As shown in [2], DT can be used to study the Dirac equation for a two–dimensional fermion in an external scalar field $w(x)$. The aim of this work is the generalization of DT for the Dirac equations with one–space–dimensional and non–stationary potentials $u(t,x)$. In Sec. 2 we show that for a fermion of transverse momenta $p \equiv p_y$, $q \equiv p_z$ the four–dimensional equation reduces to the Zakharov–Shabat equation. We demonstrate DT for this equation and the connection with intertwining and supersymmetry algebra. The result of the multiple DT (extended Crum law [3]) is set forth in Sec. 3. In Sec. 4 we show the connection between the exactly solvable Dirac (1+1) potentials and the soliton solutions of the Davey–Stewartson equations [4]. We also discuss the reduction restriction problem.

2. DARBOUX TRANSFORMATION

Let us consider the four–dimensional Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - \gamma^{\mu}A_{\mu}(t,x) + m)\Psi = 0. \tag{1}$$

We use the γ–matrix representation [2], [5]

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}, \tag{2}$$

and for $A^{\mu}(t,x)$ we have either:

$$A^{\mu}(t,x) = (0,0,A^2(t,x),A^3(t,x))^T, \tag{3}$$

or

$$A^{\mu}(t,x) = (0,0,Q(t,x),cQ(t,x))^T, \qquad c = const. \tag{4}$$

One can easily see that (3) and (4) are solutions of Maxwell equations. Let Ψ have the form:

$$\Psi = \Phi(t,x)\exp(i(py+qz)), \qquad Im\, p = Im\, q = 0. \tag{5}$$

Then Φ satisfies the Zakharov–Shabat equation:

$$\Phi_t = J\Phi_x + U\Phi. \tag{6}$$

For the case (3):

$$U = \begin{pmatrix} 0 & \tilde{A} & 0 & m-\tilde{B} \\ \tilde{A} & 0 & \tilde{B}-m & 0 \\ 0 & \tilde{B}+m & 0 & \tilde{A} \\ -\tilde{B}-m & 0 & \tilde{A} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{7}$$

where $\tilde{A} = i(A^3 - q)$, $\tilde{B} = i(p - A^2)$.

For the case (4) we get:

$$\Phi = \begin{pmatrix} A\Gamma \\ B\Gamma \end{pmatrix}, \tag{8}$$

where

$$A = \begin{pmatrix} 0 & \alpha \\ \mu & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \beta \\ \mu\rho & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \psi(t,x) \\ \phi(t,x) \end{pmatrix},$$

$$c = \frac{iq(\alpha^2+\beta^2) + 2m\alpha\beta}{ip(\alpha^2+\beta^2) + m(\alpha^2-\beta^2)}, \qquad \rho = \frac{i(\alpha p + \beta q) + \alpha m}{i(\alpha q - \beta p) + \beta m}. \tag{9}$$

The condition $c^* = c$ gives us for $\alpha \equiv \alpha_R + i\alpha_I$, $\beta \equiv \beta_R + i\beta_I$

$$\frac{\beta_I}{\alpha_I} = \frac{q\alpha_R - p\beta_R}{p\alpha_R + q\beta_R}, \qquad (p\alpha_R + q\beta_R)^2({\alpha_R}^2 - {\alpha_I}^2 + {\beta_R}^2) - {\alpha_I}^2(q\alpha_R - p\beta_R)^2 +$$

$$+2m\alpha_I({\beta_R}^2 + {\alpha_R}^2)(p\alpha_R + q\beta_R) = 0. \tag{10}$$

One of the nontrivial solutions of (10) is

$$\beta_I = 0, \qquad \beta_R = \frac{q\alpha_R}{p}, \qquad {\alpha_I}^{\pm} = \frac{\alpha_R}{p}(m \pm \sqrt{m^2+p^2+q^2}). \tag{11}$$

Substituting (8), (9) into (1) and taking (10) into account, we get a 2×2 equation (6) for Γ where $J = \sigma_3$,

$$U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$$u(t,x) = \lambda_1 + \lambda_2 Q(t,x), \qquad v(t,x) = \nu_1 + \nu_2 Q(t,x),$$

$$\lambda_1 = \frac{m\beta + i(q\alpha - p\beta)}{\mu}, \qquad \lambda_2 = \frac{1}{\mu}\frac{(\alpha^2+\beta^2)(q\alpha - p\beta - im\beta)}{ip(\alpha^2+\beta^2) + m(\alpha^2-\beta^2)},$$

$$\nu_1 = \frac{\mu(m^2+q^2+p^2)}{i(p\beta - q\alpha) - m\beta}, \qquad \nu_2 = \frac{i\mu}{\beta}(1 - c\rho). \tag{12}$$

Let χ_1 and χ_2 be 4×4 (for the case (3)) or 2×2 (for the case (4)) matrix solutions of equation (6). We define a matrix function $\tau_1 \equiv \chi_{1,x}\chi_1^{-1}$. It easy to see that τ_1 satisfies the following nonlinear equations:

$$\tau_{1,t} = \sigma_3\tau_{1,x} + [U, \tau_1] + [\sigma_3, \tau_1] + U_x. \tag{13}$$

Equation (6) is covariant with respect to DT:

$$\chi_2[1] = \chi_{2,x} - \tau_1\chi_2, \qquad U[1] = U + [\sigma_3, \tau_1]. \tag{14}$$

It is necessary to choose the function χ_1 in such a way that the structure of the matrix $U[1]$ be the same as the structure of the matrix U. This is the condition that we call the reduction restriction (see Sec.4).

The transformation (14) allows us to constract a superalgebra, in just the same way as the DT for the steady–state Shrödinger equation in a one–dimensional case [6]. In order to do this we introduce the following operators:

$$G^{(+)} = \frac{\partial}{\partial x} - \tau_1, \qquad G^{(-)} = \frac{\partial}{\partial x} + \tau_1^+. \tag{15}$$

Let us define new several operators as follow:

$$h \equiv G^{(-)}G^{(+)}, \qquad h[1] \equiv G^{(+)}G^{(-)}, \tag{16}$$

$$T \equiv \frac{\partial}{\partial t} - J\frac{\partial}{\partial x} - U, \qquad T[1] \equiv \frac{\partial}{\partial t} - J\frac{\partial}{\partial x} - U[1]. \tag{17}$$

It easy to see that

$$G^{(+)}T = T[1]G^{(+)}, \qquad TG^{(-)} = G^{(-)}T[1], \qquad [h, T] = [h[1], T[1]] = 0. \tag{18}$$

The operators h and $h[1]$ are the tipical one–dimensional matrix Hamiltonians:

$$h = \frac{\partial^2}{\partial x^2} + \rho_D\frac{\partial}{\partial x} + V, \qquad h[1] = \frac{\partial^2}{\partial x^2} + \rho_D\frac{\partial}{\partial x} + V[1], \tag{19}$$

$$\rho_D = (\tau_1^+ - \tau_1)_D, \qquad V = -(\tau_{1,x} + \tau_1^+\tau_1), \qquad V[1] = \tau_{1,x}^+ - \tau_1\tau_1^+, \tag{20}$$

where $(\tau_1)_D$ – diagonal part of τ_1. It easy to see that the operators $q^{(\pm)}$, H

$$q^{(+)} = \begin{pmatrix} 0 & 0 \\ G^{(+)} & 0 \end{pmatrix}, \qquad q^{(-)} = \begin{pmatrix} 0 & G^{(-)} \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} h & 0 \\ 0 & h[1] \end{pmatrix} \tag{21}$$

generate a supersymmetry algebra:

$$\{q^{(\pm)}, q^{(\pm)}\} = [q^{(\pm)}, H] = 0, \qquad \{q^{(+)}, q^{(-)}\} = H. \tag{22}$$

Furthermore we restrict ourselves to studing the case (4). The case (3) will be considered in a separate work.

3. EXTENDED CRUM LAW

Let us consider $2N+1$ particular solutions of (6) with $\Gamma_k \equiv (\psi_k, \phi_k)^T$, $k \leq 2N$, $\Phi \equiv (\psi, \phi)$:

$$\psi_{k,t} + \psi_{k,x} + u(t,x)\phi_k = 0, \qquad \phi_{k,t} - \phi_{k,x} + v(t,x)\psi_k = 0. \tag{23}$$

The following theorem is established:

THEOREM

Functions $\psi[N]$, $\phi[N]$ satisfy (23) with potentials $u[N]$ and $v[N]$ such that:

$$\psi[N] = \frac{\Delta_1}{D}, \qquad \phi[N] = \frac{\Delta_2}{D}, \qquad u[N] = u + 2\frac{D_1}{D}, \qquad v[N] = v - 2\frac{D_2}{D}, \tag{24}$$

where $\Delta_{1,2}$, $D_{1,2}$, D are the following determinats $(\psi^{(N)} \equiv \frac{\partial^N \psi(t,x)}{\partial x^N})$:

$$D = \begin{vmatrix} \psi_1^{(N-1)} & ... & \psi_1 & \phi_1^{(N-1)} & ... & \phi_1 \\ \psi_2^{(N-1)} & ... & \psi_2 & \phi_2^{(N-1)} & ... & \phi_2 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \psi_{2N}^{(N-1)} & ... & \psi_{2N} & \phi_{2N}^{(N-1)} & ... & \phi_{2N} \end{vmatrix},$$

$$D_1 = \begin{vmatrix} \psi_1^{(N)} & ... & \psi_1 & \phi_1^{(N-2)} & ... & \phi_1 \\ \psi_2^{(N)} & ... & \psi_2 & \phi_2^{(N-2)} & ... & \phi_2 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \psi_{2N}^{(N)} & ... & \psi_{2N} & \phi_{2N}^{(N-2)} & ... & \phi_{2N} \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \psi_1^{(N-2)} & ... & \psi_1 & \phi_1^{(N)} & ... & \phi_1 \\ \psi_2^{(N-2)} & ... & \psi_2 & \phi_2^{(N)} & ... & \phi_2 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \psi_{2N}^{(N-2)} & ... & \psi_{2N} & \phi_{2N}^{(N)} & ... & \phi_{2N} \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} \psi^{(N)} & ... & \psi & \phi^{(N-1)} & ... & \phi \\ \psi_1^{(N)} & ... & \psi_1 & \phi_1^{(N-1)} & ... & \phi_1 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \psi_{2N}^{(N)} & ... & \psi_{2N} & \phi_{2N}^{(N-1)} & ... & \phi_{2N} \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \phi^{(N)} & \psi^{(N-1)}) & ... & \psi & \phi^{(N-1)} & ... & \phi \\ \phi_1^{(N)} & \psi_1^{(N-1)}) & ... & \psi_1 & \phi_1^{(N-1)} & ... & \phi_1 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ \phi_{2N}^{(N)} & \psi_{2N}^{(N-1)}) & ... & \psi_{2N} & \phi_{2N}^{(N-1)} & ... & \phi_{2N} \end{vmatrix}. \tag{24}$$

To prove this theorem we construct N 2×2 matrix funcitons $\chi_k = (\Phi_{2k-1}, \Phi_{2k})$, $1 \leq k \leq N$:

$$\chi_k = \begin{pmatrix} \psi_{2k-1} & \psi_{2k} \\ \phi_{2k-1} & \phi_{2k} \end{pmatrix}.$$

These functions satisfy the matrix equation (6) with $J=\sigma_3$.

After N–time DT (14) we get $\chi[N]$ and $U[N]$ which satisfy (6). Let us write $\chi[N]$ as a series:

$$\chi[N]=\chi^{(N)}-\sum_{i=1}^{N}A_i(t,x)\chi^{(N-i)},\qquad A_i=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}. \tag{25}$$

Plugging (25) into (23) we get (here $C_N^k=\frac{N!}{k!\,(N-k)!}$):

$$\sum_{k=0}^{N}C_N^kU^{(k)}\chi^{(N-k)}+\sum_{k=1}^{N}[(A_{k,t}-\sigma_3A_{k,x})\chi^{(N-k)}-2\sigma_3(A_k)_{F}\chi^{(N-k+1)}]+$$

$$+\sum_{k=1}^{N}\sum_{i=0}^{N-k}C_N^iA_kU^{(i)}\chi^{(N-k-i)}-U[N](\chi^{(N)}+\sum_{k=1}^{N}A_k\chi^{(N-k)})=0, \tag{26}$$

where $(A_k)_{F}$ is the off–diagonal part of A_k. Therefore

$$u[N]=u+2b_1,\qquad v[N]=v-2c_1. \tag{27}$$

To compute b_1 and c_1 we take into account that $\chi_k[N]=0$ if $k\le N$, therefore we get a system of $2N$ equations as follows:

$$\psi_i^{(N)}=\sum_{n=1}^{N}(a_n\psi_i^{(N-n)}+b_n\phi_i^{(N-n)}),\qquad \phi_i^{(N)}=\sum_{n=1}^{N}(c_n\psi_i^{(N-n)}+d_n\phi_i^{(N-n)}), \tag{28}$$

$i=1,...,N$. Using Kramer's formulae, we get (24).

If (see(12)) $\mu=i(\beta^*-c\alpha^*)^{-1}$ then $Im\,\nu_2=Im\,\lambda_2=0$. The constants λ_1 and ν_1 may be annihilated by the standard gauge $U(1)$ transformations. Using freedom in our choice of parameters, let us assume that $v(t,x)=\kappa u(t,x),\kappa=\pm1$. Now we can supplement (24) with the transformations law for χ_1 $(N=1)$:

$$\psi_1[1]=\frac{\phi_2}{\Delta},\qquad \psi_2[1]=\frac{\kappa\phi_1}{\Delta},\qquad \phi_1[1]=\frac{\kappa\psi_2}{\Delta},$$

$$\phi_2[1]=\frac{\psi_1}{\Delta},\qquad \Delta=\begin{vmatrix}\psi_1 & \psi_2\\ \phi_1 & \phi_2\end{vmatrix}. \tag{29}$$

It is necessary to require that after N–time DT the following reduction restriction will be true:

$$v[N]=\kappa u[N]. \tag{30}$$

In the general case we do not have an algorithm allowing us to keep (30) in all the steps of DT. However, it is possible by the introduction of the so called *binary DT* which allows one to preserve the reduction restriction (30) [7].

Let us consider a closed 1–form

$$d\Omega=dx\,\zeta\chi+dt\,\zeta\sigma_3\chi,\qquad \Omega\equiv\int d\Omega \tag{31}$$

where a 2×2 matrix function ζ solves the equation:

$$\zeta_t=\zeta_x\sigma_3-\zeta U. \tag{32}$$

We shall apply the DT for (6). One can verify by immediate substitution that (32) is covariant with respect to the transform if

$$\zeta[+1] = \Omega(\zeta, \chi)\chi^{-1}. \tag{33}$$

Now we can alternatively affect U, by the following transformaton:

$$U[+1, -1] = U + [\sigma_3, \chi\Omega^{-1}\zeta]. \tag{34}$$

It may be shown that

$$\chi[+N, -N] = \chi - \sum_{k=1}^{N} \theta_k \Omega(\zeta_k, \chi), \qquad \zeta[+N, -N] = \zeta - \sum_{k=1}^{N} \Omega(\zeta, \chi_k) s_k, \tag{35}$$

where θ_k and s_k may be found from the following equations:

$$\sum_{k=1}^{N} \theta_k \Omega(\zeta_k, \chi_i) = \chi_i, \qquad \sum_{k=1}^{N} \Omega(\zeta_i, \chi_k) s_k = \zeta_i. \tag{36}$$

The transformation:

$$U[+N, -N] = U + \sum_{i,k=1}^{N} [\sigma_3, \theta_i \Omega(\zeta_k, \chi_i) s_k] \tag{37}$$

is the forementioned *binary* DT.

Let $v = \kappa u^*$ (see Sec. 4; in this section u and v are real so the condition is equivalent to (30)), then $U[+N, -N]$ will satisfy the reduction restriction if we choose ζ_k and χ_k such that:

$$\zeta_k = \chi_k R, \qquad R = diag(1, -\kappa). \tag{38}$$

The solution that follows from (34) has the form

$$u[+1, -1] = u + \frac{2\kappa(\psi_2 \phi_1^* \theta_{12}^* + \psi_1 \phi_2^* \theta_{12} - \psi_1 \phi_1^* \theta_{22} - \psi_2 \phi_2^* \theta_{11})}{\theta_{11}\theta_{22} - \mid \theta_{12} \mid^2}, \tag{39}$$

$$\theta_{ik} = \int dx\, (\psi_i^* \psi_k - \kappa \phi_i^* \phi_k) + dt\, (\psi_i^* \psi_k + \kappa \phi_i^* \phi_k).$$

Note that the square of the absolute value $u[+1, -1]$ is expressed by the compact formula:

$$\mid u[+1, -1] \mid^2 = \mid u \mid^2 - \kappa\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \ln(\theta_{11}\theta_{22} - \mid \theta_{12} \mid^2). \tag{40}$$

DT allow us to constract a rich set of the exact solutions of the Dirac equations with (1+1) potentials. In conclusion of this section we consider decreasing in $t, x \to \pm\infty$ fields (adiabatic engaging and turning-off). Let $u = v = 0$. The particular solutions of (23) are:

$$\psi_k = A_k\, e^{\omega(t-x)} + B_k\, e^{\omega(x-t)}, \qquad \phi_k = C_k\, e^{\lambda(t+x)} + D_k\, e^{-\lambda(t+x)}, \tag{41}$$

where $k = 1, 2$; A, B, C, D, ω and λ are real constants. After DT (14) under condition (12) we get:

$$Q(t, x) = \frac{1}{\mu_1 \cosh(at + bx + \delta_1) + \mu_2 \cosh(bt + ax + \delta_2)}, \tag{42}$$

$$u(t,x) = \xi\, Q(t,x), \qquad v(t,x) = -\frac{1+c^2}{\xi^2} u(t,x), \tag{43}$$

$$\xi = \mid \alpha c - \beta \mid^2 = 4\omega(A_1B_2 - A_2B_1),\ a = \omega + \lambda,\ b = \lambda - \omega,\ \mu(\beta^* - c\alpha^*) = i,$$

$$1 + c^2 = 4\lambda\xi(C_2D_1 - C_1D_2), \qquad \omega(A_1B_2 - A_2B_1) = \lambda(D_1C_2 - D_2C_1),$$

$$\mu_1 = \sqrt{(A_1C_2 - A_2C_1)(B_1D_2 - B_2D_1)},\ \mu_2 = \sqrt{(B_1C_2 - B_2C_1)(A_1D_2 - A_2D_1)},$$

$$\delta_1 = \frac{1}{2}\ln\frac{A_1C_2 - A_2C_1}{B_1D_2 - B_2D_1}, \qquad \delta_2 = \frac{1}{2}\ln\frac{B_1C_2 - B_2C_1}{A_1D_2 - A_2D_1},$$

where α, β satisfy (11). The potential (42) is a localized impulse, which decreases in $t \to \pm\infty$. A particular solution of the Dirac equation may be calculated using (5), (8) from $(\psi_{1,2}[1], \phi_{1,2}[1])^T$, where $\psi_{1,2}[1]$ and $\phi_{1,2}[1]$ are defined by (29). This bispinor describes the quazi–stationary state of a fermion. The fermion is free along the y and z and restrained along the x–axis.

4. DAVEY-STEWARTSON EQUATIONS.

It is shown in [2] that a class of the exactly soluble Dirac one–dimensional potentials corresponds to a soliton solutions of the MKdV equation, just as certain Schrödinger potentials are solutions of the KdV equation. In this section we show that there exist the analogous connection between exactly soluble Dirac (1+1) potentials and soliton solutions of the Davey–Stewartson equations (DS).

The DS equations appear as the commutation condition of the two operators [7], [8]:

$$[T_1, T_2] = 0, \tag{44}$$

where $T_1 \equiv T$ (see (17)) and T_2 is given by the formula:

$$T_2 = i\frac{\partial}{\partial y} + 2\sigma_3\frac{\partial^2}{\partial x^2} + 2U\frac{\partial}{\partial x} + U_x + \sigma_3 U_t + A, \tag{45}$$

$$A = diag(A_1, A_2), \qquad v = \kappa u^*,$$

$$A_1 = -\kappa \mid u \mid^2 + \frac{1}{2i} l_{(+)} F, \qquad A_2 = \kappa \mid u \mid^2 + \frac{1}{2i} l_{(-)} F.$$

Here $l_{(\pm)} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}$ and F is a pure imaginary function. The operator T_2 is also covariant with respect to DT, what allows one to get infinite sets of exact solutions of DS, for example soliton solutions, exponentially localized on the plane – the dromions (42) [7], [8]. The DS equations contain two fields: $u(y,t,x)$ – the amplitude wavetrain function and $S(y,t,x) \equiv -iF_x$ – the amplitude function of slowly changing in t (space variable!) and x mean field:

$$iu_y + u_{xx} + u_{tt} - 2\kappa \mid u \mid^2 u + Su = 0, \qquad l_{(+)} l_{(-)} S = -4\kappa(\mid u \mid^2)_{xx}. \tag{46}$$

The dromions S–component is constant along the two ortogonal directions and moves with the constant velocity [8].

In Sec.3 we have shown that the one–dromion solution corresponds to adiabatic engaging and turning-off electro-magnetic field. Boity, Leon, Martina and Pempinelli [8] showed that two dromions scatter against each other with the soliton phase shift, therefore the N–dromions solution corresponds to a adiabatic external field, too. A many–dromion solution may be obtained by the N–time DT. For this purpose we transform the LA–pair (T_1 and T_2).

Introducing new variables $x = p+q$ and $t = p-q$ we exclude a field F from (45). The matrix A takes the form

$$A = \begin{pmatrix} \kappa \int_{-\infty}^{q} dq\, \frac{\partial |u|^2}{\partial p} + g_1(p,y) & 0 \\ 0 & \kappa \int_{-\infty}^{p} dp\, \frac{\partial |u|^2}{\partial q} + g_2(q,y) \end{pmatrix}, \tag{47}$$

where $g_1(p,y)$ and $g_2(q,y)$ are arbitrary functions. We suppose that $u(p,q,y)$ belongs to a Schwarz space L. Then a nonlocal flow is given as follows:

$$S = 2\kappa \mid u \mid^2 + (\int_{-\infty}^{q} dq\, \frac{\partial}{\partial p} + \int_{-\infty}^{p} dp\, \frac{\partial}{\partial q}) \mid u \mid^2 + g_1 - g_2. \tag{48}$$

In the new variables

$$T_1 = \begin{pmatrix} -\frac{\partial}{\partial q} & u \\ \kappa u^* & \frac{\partial}{\partial p} \end{pmatrix}, \tag{49}$$

therefore for $u \in L$, the equation $T_1 \chi = 0$ gives at infinity for components of χ the following condition:

$$\psi_k = \psi_k(p,y), \qquad \phi_k = \phi_k(q,y), \qquad k = 1,2. \tag{50}$$

The second equation $T_2 \chi = 0$ gives four nonstationary one–dimensional Schrödinger equations

$$i\psi_{k,y} + \frac{1}{2}\psi_{k,pp} + v_1 \psi_k = 0 \tag{51}$$

$$i\phi_{k,y} - \frac{1}{2}\phi_{k,qq} + v_2 \phi_k = 0 \tag{52}$$

where

$$v_1(p,y) = g_1 + \kappa \int_{-\infty}^{+\infty} dq\, \frac{\partial \mid u \mid^2}{\partial p}, \qquad v_2(q,y) = g_2 + \kappa \int_{-\infty}^{+\infty} dp\, \frac{\partial \mid u \mid^2}{\partial q}. \tag{53}$$

For the case of a dromion (42) we have

$$v_1 = 4\omega^2 sec^2 f_1, \qquad v_2 = -4\lambda^2 sec^2 f_2,$$

$$f_1 = 2\omega(p - 2ay) + \frac{1}{2} \ln \frac{A_0}{b_0}, \qquad f_2 = 2\lambda(q + 2by) + \frac{1}{2} \ln \frac{C_0}{D_0}, \tag{54}$$

where a, b, $(A,B,C,D)_0$ are real constants. So the dromion sits on two plane solitons that correspond to the one–level reflectionless potentials. They may be easily obtained by the standard DT for the Schrödinger equation [1] on zero background:

$$\psi_k[1] = \frac{\frac{\partial \psi_k}{\partial p}\psi_0 - \frac{\partial \psi_0}{\partial p}\psi_k}{\psi_0}, \quad \phi_k[1] = \frac{\frac{\partial \phi_k}{\partial q}\phi_0 - \frac{\partial \phi_0}{\partial q}\phi_k}{\phi_0},$$

$$v_1[1] = -2\frac{\partial^2}{\partial p^2} \ln \psi_0, \qquad v_2[1] = 2\frac{\partial^2}{\partial q^2} \ln \phi_0. \tag{55}$$

The background eigenfunctions are choosen such that:

$$\psi_0(p,y) = 2\sqrt{A_0 B_0}\, \cosh f_1\, e^{i\theta}, \tag{56}$$

$$\phi_0(q,y) = 2\sqrt{C_0 D_0}\, \cosh f_2\, e^{i\vartheta}, \tag{57}$$

where

$$\theta = 2[(\omega^2 - a^2)y + ap], \qquad \vartheta = 2[(b^2 - \lambda^2)y + bq], \tag{58}$$

and the new coefficients $(A,B,C,D)_0$ satisfy the conditions:

$$A_0(B_1C_2 - B_2C_1) = B_0(A_1C_2 - A_2C_1), \qquad C_0(A_1D_2 - A_2D_1) = D_0(B_1C_2 - B_2C_1).$$

For the second DT it is necessary to switch to the two–level potentials in equations (51), (52). Let $\varpi > \omega$, $\varrho > \lambda$ then two "linearly independent" solutions that correspond to the eigenvalues 2ϖ and 2ϱ have the form:

$$\psi_{-1}^{(+)} = A_{-1}\, e^{2\varpi(p-2\alpha y)+i\theta_{-1}}, \qquad \psi_{-1}^{(-)} = B_{-1}\, e^{-2\varpi(p-2\alpha y)+i\theta_{-1}} \tag{59}$$

for (51) and

$$\phi_{-1}^{(+)} = C_{-1}\, e^{2\varrho(q+2\beta y)+i\vartheta_{-1}}, \qquad \phi_{-1}^{(-)} = D_{-1}\, e^{-2\varrho(q+2\beta y)+i\vartheta_{-1}} \tag{60}$$

for (52), where θ_{-1} and ϑ_{-1} differ from θ and ϑ by substitution $(a,b,\lambda,\omega) \to (\alpha,\beta,\varrho,\varpi)$. We transform $\psi_{-1}^{(\pm)}$ and $\phi_{-1}^{(\pm)}$ by the formula (55) with ψ_0 and ϕ_0 accordingly. Then we will build the new support function:

$$\psi_{-1}[1] = \psi_{-1}^{(+)}[1] - \psi_{-1}^{(-)}[1], \qquad \phi_{-1}[1] = \phi_{-1}^{(+)}[1] - \phi_{-1}^{(-)}[1]. \tag{61}$$

For simplicity let us choose $A = B = C = D = 1$. As a result we have:

$$\psi_{-1}[1] = (\frac{(\varpi-\omega)\cosh(\xi_0+\xi_1) + (\varpi+\omega)\cosh(\xi_0-\xi_1)}{\cosh \xi_0} + i(\alpha - a)\sinh \xi_0)e^{i\theta_{-1}}, \tag{62}$$

$$\phi_{-1}[1] = (\frac{(\varrho-\lambda)\cosh(\eta_0+\eta_1) + (\varrho+\lambda)\cosh(\eta_0-\eta_1)}{\cosh \eta_0} + i(\beta - b)\sinh \eta_0)e^{i\vartheta_{-1}}, \tag{63}$$

where $\xi_0 = 2\omega(p-2ay), \xi_1 = 2\varpi(p-2ay), \eta_0 = 2\lambda(q+2by), \eta_1 = 2\varrho(q+2by)$. If one puts $\alpha = a$, $\beta = b$ then (62), (63) exactly coincide with the supporting functions that generate the two–level potential with respect to the Darboux transformations [8].

Now it is necessary to define four functions $\psi_k[1]$, $\phi_k[1]$ $(k = 1,2)$ which are the solutions of (51), (52) with the potentials (54). The basic problem at this step is the independence of the wronskians

$$\begin{vmatrix} \frac{\partial\psi_2}{\partial p} & \frac{\partial\psi_1}{\partial p} \\ \psi_2 & \psi_1 \end{vmatrix}, \qquad \begin{vmatrix} \frac{\partial\phi_2}{\partial q} & \frac{\partial\phi_1}{\partial q} \\ \phi_2 & \phi_1 \end{vmatrix},$$

with respect to the space coordinates, what is necessary for the solvability of the reduction restriction equation $v[2] = \kappa\, u^*[2]$. For this purpose it is enough to choose:

$$\psi_k[1] = A_{k,-1}(\varpi - \omega\, \tanh \xi_0)\psi_{-1}^{(+)} + B_{k,-1}(\varpi + \omega\, \tanh \xi_0)\psi_{-1}^{(-)}, \tag{64}$$

$$\phi_k[1] = C_{k,-1}(\varrho - \lambda\, \tanh \eta_0)\phi_{-1}^{(+)} + D_{k,-1}(\varrho + \lambda\, \tanh \eta_0)\phi_{-1}^{(-)}, \tag{65}$$

with the constants that satisfy the relation:

$$\varpi(\omega^2 - \varpi^2)(B_{1,-1}A_{2,-1} - B_{2,-1}A_{1,-1}) = \kappa\, \varrho(\lambda^2 - \varrho^2)(C_{1,-1}D_{2,-1} - C_{2,-1}D_{1,-1}). \tag{66}$$

Therefore we choose $\psi_{-2}[2]$, $\phi_{-2}[2]$ as the wave functions that generate a three–level potential (via DT) and determine the corresponding functions $\psi_k[2]$ and $\phi_k[2]$. One may repeat this procedure and realize the third DT.

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